# Parametric Problems in Power System Analysis: Recent Applications of Polynomial Approximation Based on Galerkin Method

Hao Wu, Danfeng Shen, Bingqing Xia, Yiwei Qiu, Yongzhi Zhou, and Yonghua Song

***Abstract*—In power systems, there are many uncertainty factors such as power outputs of distributed generations and fluctuations of loads. It is very beneficial to power system analysis to acquire an explicit function describing the relationship between these factors (namely parameters) and power system states (or performances). This problem, termed as parametric problem (PP) in this paper, can be solved by Galerkin method, which is a powerful and mathematically rigorous method aiming to seek an accurate explicit approximate function by projection techniques. This paper provides a review of the applications of polynomial approximation based on Galerkin method in power system PPs as well as stochastic problems. First, the fundamentals of polynomial approximation and Galerkin method are introduced. Then, the process of solving three types of typical PPs by polynomial approximation based on Galerkin method is elaborated. Finally, some application examples as well as several potential applications of power system PPs solved by Galerkin method are presented, namely the probabilistic power flow, approximation of static voltage stability region boundary, and parametric time-domain dynamic simulation.**

***Index Terms*—Parametric problem, stochastic problem, power system analysis, polynomial approximation, Galerkin method.**

## I. Introduction

POWER system analysis is often confronted with a kind of problems that concern the impact of some uncertainty factors on system states or performances [1], for example, the impact of penetration level of distributed generations (DGs) on system performance [2], and the impact of bilateral contract on power transaction, the impact of load level on system stability [3]. These factors, albeit with different physical meanings, can all be mathematically regarded as parameters of power system analysis model. Accordingly, system performances can be regarded as system states.

Hence the problems at hand, termed parametric problems (PPs) in this paper, are to seek an explicit function describing the relationship between parameters and states [4]. Since the parameter-state relationship in power system analysis is usually implicit and too complicated to obtain exactly, the PP comes down to acquire an accurate explicit approximate function that describes this relationship.

Sensitivity methods and sampling-fitting methods are the most widely used techniques for PPs in power system analysis. Sensitivity methods [5] aim to analyze the sensitivity of system states (or outputs) to system parameters (or inputs), and are generally associated with linearizing the parameter-state relationship around the rated parameter value. If the parameters vary in a small range or the system possesses a good linear characteristic, the sensitivity methods can result in a linear approximate function with satisfactory accuracy. Otherwise, the resulting linear function may have poor accuracy, and high-order sensitivity methods [6] should be adopted to retain the accuracy. However, the derivation of high-order sensitivity is tedious and practically difficult to perform, when the expansion order is greater than two or the parameter number is more than two. Anyway, sensitivity methods are local since it is usually based on the Taylor series expansion technique. Therefore, even high-order sensitivity methods will lose accuracy when the system has strong nonlinearity in the variation range of parameters.

Sampling-fitting methods [7], [8] aim to construct an approximate function by many samples. Their basic process is to, first sample the parameters in their variation ranges, then acquire the values of parameter-state function at all sampling points of parameters, and finally employ the fitting technique to get an approximate function. Sampling-fitting methods provide a global description of the parameter-state relationship over the whole parameter variation ranges, and thus are preferable in the case of strong nonlinearity. However, they have defects in terms of high dimension and accuracy control, i.e., high dimensional PPs require so many samples that this method may be computationally prohibitive, and the sample size is hard to be determined in accordance with the accuracy requirement.

An important special case of PPs is the stochastic problem [9], in which parameters additionally follow probability dis-

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tributions. The stochastic problem aims to calculate the probability uncertainty of system states caused by the probability uncertainty of parameters. The acquired probability distributions and statistical characteristics of system states are valuable for many power system analyses such as risk assessments of system planning and operation [10], [11].

Typical methods for the stochastic problem include Monte Carlo simulation (MCS) [12], cumulant-based methods (CMs) [13], and point estimation methods (PEMs) [14]. MCS, based on random sampling, is accurate and flexible but quite time-consuming, and thus it often serves as the benchmark for other methods. CMs require a linearization of the parameter-state relationship just like the sensitivity methods, and therefore they are fast but inaccurate in the case of strong nonlinearity. PEMs are based on deterministic sampling technique and can accurately calculate the low-order moments of system states but may fail to obtain accurate high-order moments.

This paper provides a review of solving power system PPs and stochastic problems by a new method called the polynomial approximation based on Galerkin method [15], [16]. Polynomial approximation [17]-[19], as a subclass of function approximation [20], concerns how to accurately and globally approximate a function with a simple polynomial. It is a popular topic in the field of computational mathematics, and has wide applications in physical and engineering problems. Two typical methods for polynomial approximation are collocation methods and Galerkin method.

Collocation methods, usually based on numerical integration [21] or interpolation [22] technique, construct the polynomial approximation by acquiring values of the parameter-state function at some collocation points. They are somewhat similar to the sampling-fitting methods, but the points are deliberately determined by certain technique (usually zeros of orthogonal polynomials) rather than come from previously-given data, and the construction process usually involves some advanced techniques such as Smolyak sparse grid [23]. Collocation methods are easy to realize, but are not necessarily accurate because they are not theoretically rigorous.

Galerkin method [16], [24]-[27] aims to find an explicit approximate function for the implicit parameter-state function by projecting system model equations containing parameters onto an approximation space. The projection process is so mathematically rigorous that the acquired approximate function is very accurate. By combining polynomial approximation and Galerkin method, the approximate polynomial solution to PPs is obtained, which can also further tackle the stochastic problems. We have done much work on solving power system PPs by the polynomial approximation based on Galerkin method [4], [28]-[34], and hence present this review.

This paper is organized as follows. Section II narrates the concept of PPs and presents three types of typical PPs in power system analysis from the perspective of mathematical models. Section III provides the fundamentals of polynomial approximation and Galerkin method. Section IV elaborates how to solve PPs by polynomial approximation based on Galerkin method. Section V introduces four application examples and several potential applications of power system PPs solved by Galerkin method. Section VI concludes the paper and briefly introduces prospective worthwhile works.

## II. PPs in Power System Analysis

### A. Concept of PPs

The concept of PPs provides a unified perspective for a category of problems of studying the impacts of some uncertainty factors (namely parameters) on system states or performances.

The parameters in PPs are a general concept and could be arbitrary quantifiable impact factors of the system. They are not limited to operation or control variables such as the load power and the terminal voltage of generator, but could also be system model quantities such as cost coefficients of generators and rotational inertia of generator units, and uncontrollable stochastic factors such as power outputs of DG.

The model of PPs includes both the system state equations and the system output equations that define system performances. For the convenience of narration, system performances are treated as states in PPs. In this way, PP model contains only two types of variables, i.e., states and parameters, and thus defines an implicit function relationship between them.

The aim of PPs is to seek an explicit function that describes the relationship between parameters and states. With this explicit function, the subsequent analysis on the characteristics of the system will become much easier, which is the main significance of PPs. The parameters and states can be continuous or discrete. In order to focus on PPs, this paper only considers the continuous-variable case, which is the most common case of PPs.

Besides being deterministic in certain ranges, the parameters can also be stochastic, i.e., random variables following certain probability distributions or stochastic processes. The resulting problem is called the stochastic problem, which can be regarded as a special case of PPs and hence solved by PP methods in combination with some extra probabilistic and statistical manipulations.

There are a variety of PPs in power system analysis. From the perspective of mathematical models, most of them can be classified into the following three types of typical PPs, as illustrated by Fig. 1.

### B. Parametric Nonlinear Algebraic Equation (NAE) Problems

The simplest form of PPs is parametric NAE problems, which are modeled as:

$$\mathbf{0}=\boldsymbol{f}(\boldsymbol{x},\boldsymbol{p}) \tag{1}$$

where $\boldsymbol{x}=[x_1,x_2,\ldots,x_n]\in\mathbb{R}^n$ and $\boldsymbol{p}=[p_1,p_2,\ldots,p_l]\in\mathbb{R}^l$ are the vectors of state variables and parameters, respectively; $n$ and $l$ are the dimensions of $\boldsymbol{x}$ and $\boldsymbol{p}$, respectively; and $\boldsymbol{f}(\cdot):\mathbb{R}^n\times\mathbb{R}^l\rightarrow\mathbb{R}^n$ is the vector of nonlinear functions.

The goal of the problem is to find an explicit function $\boldsymbol{x}^*(\boldsymbol{p})$ that satisfies (1) and hence gives the relationship between $\boldsymbol{x}$ and $\boldsymbol{p}$. However, the exact explicit function $\boldsymbol{x}^*(\boldsymbol{p})$ rarely exists in practical engineering problems, and thus an approximate explicit function $\boldsymbol{x}(\boldsymbol{p})$ is the only feasible choice.

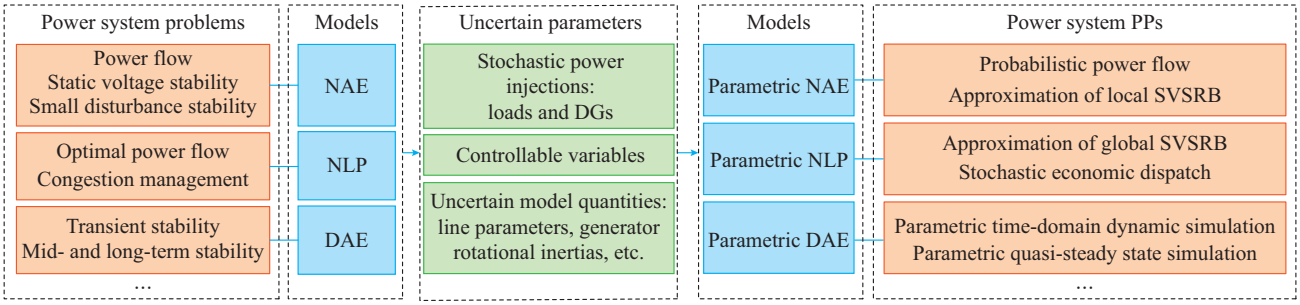


Fig. 1. Classification and examples of PPs in power system analysis.

There are many methods for obtaining the approximate explicit function, e.g., sensitivity methods and sampling-fitting methods. This paper introduces the polynomial approximation based on Galerkin method, which has the advantage of global nonlinear approximation and high accuracy. The general solving process of parametric NAE problems by the introduced method is given in Section IV-A. Since many power system problems share the form (1), this method has many applications. Two typical applications, namely probabilistic power flow (PPF) and static voltage stability region boundary (SVSRB), are presented in Section V-A and Section V-B, respectively.

### C. Parametric Nonlinear Programming (NLP) Problems

NLP has a variety of applications in power system analysis, among which optimal power flow (OPF) is one of the most important. In the case of parameter uncertainty, the optimum solution changes with these parameters. This yields the parametric NLP problems, which are modeled as:

$$\begin{cases} \min\limits_{\boldsymbol{x}} \ \boldsymbol{f}(\boldsymbol{x},\boldsymbol{p}) \\ \text{s.t.} \ \ \boldsymbol{h}(\boldsymbol{x},\boldsymbol{p})=\boldsymbol{0} \\ \qquad \boldsymbol{g}(\boldsymbol{x},\boldsymbol{p})\le\boldsymbol{0} \end{cases} \tag{2}$$

where $\boldsymbol{h}(\cdot):\mathbb{R}^n\times\mathbb{R}^l\to\mathbb{R}^r$ and $\boldsymbol{g}(\cdot):\mathbb{R}^n\times\mathbb{R}^l\to\mathbb{R}^s$ are the vector of $r$ equality constraints and vector of $s$ inequality constraints, respectively.

The optimum solution $\boldsymbol{x}_{\text{opt}}^*$ of (2) is a function of $\boldsymbol{p}$. The goal of the problem is to find an explicit function $\boldsymbol{x}_{\text{opt}}^*(\boldsymbol{p})$ describing the relationship between $\boldsymbol{x}_{\text{opt}}^*$ and $\boldsymbol{p}$. Similarly, the exact $\boldsymbol{x}_{\text{opt}}^*(\boldsymbol{p})$ hardly exists and it is unavoidable to find an approximation $\boldsymbol{x}_{\text{opt}}(\boldsymbol{p})$ as an alternative.

It is not as straight-forward as solving the algebraic problems (1) to solve the parametric NLP problems by the introduced Galerkin method. The Galerkin method should be combined with some extra optimization techniques, as introduced in Section IV-B. Parametric NLP (2) also has plenty of applications in power system analysis. A typical application called approximating the global SVSRB is presented in Section V-C.

### D. Parametric Differential-algebraic Equation (DAE) Problems

Power system dynamic problems are usually modeled as DAEs. Parametric DAE problems aim to study the relationship between parameters and system dynamics, and are modeled as:

$$\begin{cases} \dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p}) \\ \boldsymbol{0}=\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{p}) \end{cases} \tag{3}$$

where $\boldsymbol{y}=[y_1,y_2,\ldots,y_m]\in\mathbb{R}^m$ is the vector of algebraic state variables; $m$ is the dimension of $\boldsymbol{y}$; and $\boldsymbol{f}(\cdot):\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}^n$ and $\boldsymbol{g}(\cdot):\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}^m$ are the vectors of $n$ dynamic equations and $m$ algebraic equations, respectively.

Given $\boldsymbol{p}$, (3) defines trajectories $\boldsymbol{x}^*(t)$ and $\boldsymbol{y}^*(t)$ that depict system dynamics. The goal of the problem is to find explicit functions $\boldsymbol{x}^*(t,\boldsymbol{p})$ and $\boldsymbol{y}^*(t,\boldsymbol{p})$ that describe the impacts of $\boldsymbol{p}$ on system trajectories. Similarly, the exact $\boldsymbol{x}^*(t,\boldsymbol{p})$ and $\boldsymbol{y}^*(t,\boldsymbol{p})$ hardly exist, and accurate approximation $\boldsymbol{x}(t,\boldsymbol{p})$ and $\boldsymbol{y}(t,\boldsymbol{p})$ are desired.

The general solving process of the parametric DAE problems by the Galerkin method is introduced in Section IV-C. Parametric DAE problem (3) has many applications in both deterministic and stochastic power system dynamic analysis. A typical application called power system time-domain simulation considering parameter uncertainty, is presented in Section V-D.

## III. Fundamentals of Polynomial Approximation and Galerkin Method

### A. Approximation of Explicit Function

It is well known that a periodic function can be approximated by the weighted sum of sinusoidal functions according to the Fourier analysis. Likewise, a continuous function can be approximated by the weighted sum of monomials, i.e.,

$$x(p)\approx x^N(p)=\sum_{i=0}^{N}c_i p^i \tag{4}$$

where $p$ is a variable; $x(\cdot)$ is a continuous function in an inteval $[a,b]$; $x^N(\cdot)$ is the $N^{\text{th}}$-order polynomial approximation of $x(\cdot)$; and $c_i$ is the expansion coefficient of the monomial $p^i$. The rationality of (4) is guaranteed by the Weierstrass approximation theorem [35]: any continuous function in a closed interval can be uniformly approximated as closely as desired by a polynomial function, which means the infinite norm approximation error $\max\limits_{a\le p\le b}|x^N(p)-x(p)|$ could be arbitrarily small as the order $N$ increases.

$x^N(p)$ in (4) is expressed in the form of the basis $\{p^i, i=1,2,\ldots,N\}$, and thus is a function in the linear functional space (called the approximation space) spanned by this ba-

sis. According to the linear algebra theory, it can also be expressed in the form of another basis in the same space such as the polynomial basis $\{\Phi_i(p), i=0,1,\ldots,N\}$, where $\Phi_i(p)$ is an $i^{\text{th}}$ degree polynomial. Therefore, this approximation becomes

$$x^N(p)=\sum_{i=0}^{N}c_i\Phi_i(p) \tag{5}$$

Let $\boldsymbol{c}$ denote the vector composed of all coefficients $c_i$ ($i=0,1,\ldots,N$), then $\boldsymbol{c}$ can be regarded as the coordinate of $x^N(p)$ in the spanned functional space, just like a tuple indicates the coordinate of a point in the Euclidean space.

To evaluate the discrepancy between $x(p)$ and $x^N(p)$, we introduce the following residual function $R(p)$:

$$R(p)=x(p)-x^N(p)=x(p)-\sum_{i=0}^{N}c_i\Phi_i(p) \tag{6}$$

### B. Least-square Approximation

The least-square approximation is one of the most popular methods for constructing a polynomial approximation $x^N(p)$, and is optimal in the sense of weighted $L_2$ norm error.

Let $f(p)$ and $g(p)$ denote arbitrary two functions in $[a,b]$, and $w(p)$ denote a nonnegative function (or weight function, usually $w(p)=1$) in $[a,b]$. Then the weighted inner product of $f(p)$ and $g(p)$ is defined as (7), and the weighted $L_2$ norm of $f(p)$ is defined as (8).

$$\langle f(p),g(p)\rangle=\int_a^b f(p)g(p)w(p)\mathrm{d}p \tag{7}$$

$$\|f(p)\|_2=\sqrt{\langle f(p),f(p)\rangle}=\sqrt{\int_a^b f^2(p)w(p)\mathrm{d}p} \tag{8}$$

The weighted least-square approximation aims to find a group of coefficients such that the weighted $L_2$ norm of the residual function (namely weighted $L_2$ norm error) is minimized, namely

$$\min_{c}\ R=\|R(p)\|_2^2=\int_a^b\left(x(p)-\sum_{i=0}^{N}c_i\Phi_i(p)\right)^2w(p)\mathrm{d}p \tag{9}$$

At the minimum of $R$, the partial derivative of $R$ with respect to every $c_j$ equals to zero, namely

$$0=\frac{\partial R}{\partial c_j}=-2\int_a^b\left(x(p)-\sum_{i=0}^{N}c_i\Phi_i(p)\right)\Phi_j(p)w(p)\mathrm{d}p=$$
$$-2\left\langle x(p)-\sum_{i=0}^{N}c_i\Phi_i(p),\Phi_j(p)\right\rangle\quad j=0,1,\ldots,N \tag{10}$$

With a transformation, (10) becomes

$$\sum_{i=0}^{N}c_i\langle\Phi_i(p),\Phi_j(p)\rangle=\langle x(p),\Phi_j(p)\rangle\quad j=0,1,\ldots,N \tag{11}$$

Note that the inner products are with respect to $p$, and that $p$ will disappear after all inner products are worked out. Therefore, the only unknowns in (11) are $N+1$ coefficients, which can be obtained by solving this $N+1$ dimensional linear equation set (11).

Specially, if the system of polynomials $\{\Phi_i(p),\ i=0,1,\ldots,N\}$ is orthogonal with respect to $w(p)$, i.e., for any $i$ and $j$, there exists (12), then (11) becomes (13).

$$\langle\Phi_i(p),\Phi_j(p)\rangle=\begin{cases}\chi_i & i=j\\ 0 & i\neq j\end{cases} \tag{12}$$

$$c_i=\langle x(p),\Phi_i(p)\rangle/\chi_i\quad i=0,1,\ldots,N \tag{13}$$

where $\chi_i$ is a positive constant.

This means that under the circumstance of orthogonal polynomial basis, every coefficient in the polynomial approximation equals to the projection value (ignoring the scale constant $\chi_i$) of the function onto the corresponding polynomial basis term.

### C. Galerkin Method

The least-square formula (10) can also be written as:

$$\langle R(p),\Phi_j(p)\rangle=0\quad j=0,1,\ldots,N \tag{14}$$

Equation (14) reveals another perspective of understanding the least-square approximation, that is, letting the projection of $R(p)$ onto the approximation space spanned by basis $\{\Phi_j(p), j=0,1,\ldots,N\}$ be 0, as illustrated by Fig. 2.

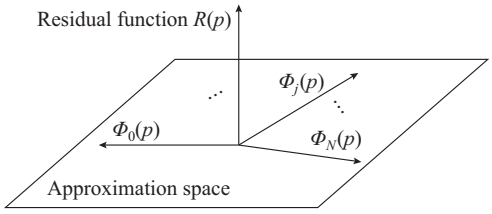


Fig. 2. Illustration of Galerkin projection.

This understanding is equivalent to the so-called Galerkin method [16], which aims to seek an explicit approximate function in the approximation space such that the residual function is orthogonal to arbitrary functions in the approximation space. Equation (12) is called the Galerkin equations, which form an $N+1$ dimensional nonlinear equation set with respect to coefficients $\{c_i, i=0,1,\ldots,N\}$. In Galerkin method, the basis $\{\Phi_i(p), i=0,1,\ldots,N\}$ in the polynomial approximation (5) is called the trial basis, and the basis $\{\Phi_j(p), j=0,1,\ldots,N\}$ used to project residual function in (14) is called the test basis.

Under the residual function definition (6), Galerkin method is identical to the least-square approximation, and thus is optimal. However, the genuine powerfulness of Galerkin method lies in that residual function $R(p)$ is not necessary to be (6), rendering this method still applicable when $x(p)$ denotes an implicit function. Under this circumstance, the least-square approximation is not feasible since the inner product on the right side of (11) cannot be worked out. In contrast, Galerkin method still works and provides a quasi-optimal approximation, if an appropriate residual function is chosen. To focus on the essence of Galerkin method, this subsection only discusses the explicit function, and the implicit function will be discussed in Section V.

The Galerkin method can be extended to a more general sense. In traditional Galerkin method, the test basis is the same as the trial basis. However, generalized Galerkin method [30] allows for using a different test basis $\{\Psi_j(p)\}$, so

(14) becomes

$$\langle R(p), \Psi_j(p)\rangle = 0 \quad j = 0, 1, \ldots, N \tag{15}$$

Compared with (14), (15) loses the link to least-square approximation, and hence the approximation accuracy of generalized Galerkin method may not be as good as that of traditional Galerkin method. But the ability of choosing a test basis different from the trial basis endows the generalized Galerkin method with some practical flexibility, which may help simplify the computation and reduce the time cost.

It should be noted that both (14) and (15) rely on the inner product operation defined by (7) or (8), which is an integration over the whole interval $[a, b]$. This endows Galerkin method with the characteristic of global approximation. That is, if approximation performance over the whole interval $[a, b]$ is concerned, and interval $[a, b]$ is large or the system has strong nonlinearity, Galerkin method usually performs better than local approximation methods such as sensitivity methods based on Taylor series expansion.

The main procedures of applying Galerkin method to polynomial approximation are shown in Fig. 3.

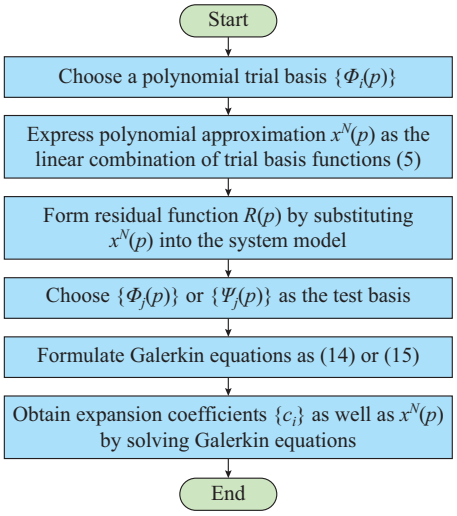


Fig. 3. Solving procedures of Galerkin method.

### *D. Galerkin Method for Approximating Multivariate Function Vector*

The Galerkin method in Section III-C can also be used to approximate multivariate function vector, as described below.

Firstly, construct the basis for multivariate polynomial approximation. For every variable $p_d (d = 1, 2, \ldots, l)$, we can build its univariate polynomial sequence $\{\Phi_{di}(p_d), i = 0, 1, \ldots\}$. By selecting a polynomial from every univariate sequence and prescribing the degree sum of the $l$ univariate polynomials no more than a given order $N$, we construct the $N^{\text{th}}$-order basis as

$$\Phi_i(\boldsymbol{p}) = \Phi_{1i_1}(p_1)\Phi_{2i_2}(p_2)\cdot\ldots\cdot\Phi_{li_l}(p_l) \quad 0 \le |\boldsymbol{i}| \le N \tag{16}$$

where $\boldsymbol{i} = [i_1, i_2, \ldots, i_l]$; and $|\boldsymbol{i}| = i_1 + i_2 + \ldots + i_l$. The size of the $N^{\text{th}}$-order basis, namely the number of constituent polynomial terms in (16), is

$$N_b = \frac{(N+l)!}{N!\,l!} \tag{17}$$

Take the 2-variable 3rd-order basis (namely $l = 2, N = 3$) for example. If adopting the simple monomial sequence for each $p_d$, i. e., $\{1, p_d, p_d^2, p_d^3\}$ $(d = 1, 2)$, the basis (16) will be $\{1, p_1, p_2, p_1^2, p_2^2, p_1 p_2, p_1^3, p_2^3, p_1^2 p_2, p_1 p_2^2\}$ and the basis size is 10.

Multi-index $\boldsymbol{i}$ is inconvenient for practical applications and hence often translated into single index $i$ orderly. The most popular translation method is the graded lexicographic order, which prescribes that $\boldsymbol{i} > \boldsymbol{j}$ if and only if $|\boldsymbol{i}| \ge |\boldsymbol{j}|$ and that the first nonzero entry in the difference $\boldsymbol{i} - \boldsymbol{j}$ is positive. By rearranging these multi-indices from smallest to largest and endowing each of them with a single index according its position, the basis $\{\Phi_i(\boldsymbol{p})\}$ $(0 \le |\boldsymbol{i}| \le N)$ can be written as $\{\Phi_i(\boldsymbol{p})\}$ $(i = 1, 2, \ldots, N_b)$.

Now consider an explicit function vector $\boldsymbol{x}(\boldsymbol{p})$, where $\boldsymbol{p} \in \mathbb{R}^l$ and $\boldsymbol{x}(\cdot): \mathbb{R}^l \to \mathbb{R}^n$. By employing the trial basis $\{\Phi_i(\boldsymbol{p})\}$ $(i = 1, 2, \ldots, N_b)$, each component $x_k(\boldsymbol{p})$ of $\boldsymbol{x}(\boldsymbol{p})$ can be approximated by

$$x_k^N(\boldsymbol{p}) = \sum_{i=1}^{N_b} c_{ki}\Phi_i(\boldsymbol{p}) \quad i = 1, 2, \ldots, n \tag{18}$$

where $c_{ki}$ $(i = 1, 2, \ldots, N_b)$ is the expansion coefficient for $x_k^N$. The vector comprised of $x_k^N(\boldsymbol{p}), k = 1, 2, \ldots, n$ is denoted by $\boldsymbol{x}^N(\boldsymbol{p})$. Similarly, the residual function $R_k(\boldsymbol{p})$ for approximating $x_k(\boldsymbol{p})$ is

$$R_k(\boldsymbol{p}) = x_k(\boldsymbol{p}) - x_k^N(\boldsymbol{p}) = x_k(\boldsymbol{p}) - \sum_{i=1}^{N_b} c_{ki}\Phi_i(\boldsymbol{p}) \tag{19}$$

In order to use Galerkin method, a trial basis should be chosen, such as the basis $\{\Phi_i(\boldsymbol{p})\}$ $(i = 1, 2, \ldots, N_b)$ or another polynomial basis $\{\Psi_j(\boldsymbol{p})\}$ $(j = 1, 2, \ldots, N_b)$. Then we have Galerkin equation (20) if traditional Galerkin method is adopted or (21) if generalized Galerkin method is adopted.

$$\langle R_k(\boldsymbol{p}), \Phi_j(\boldsymbol{p})\rangle = 0 \quad k = 1, 2, \ldots, n; j = 1, 2, \ldots, N_b \tag{20}$$

$$\langle R_k(\boldsymbol{p}), \Psi_j(\boldsymbol{p})\rangle = 0 \quad k = 1, 2, \ldots, n; j = 1, 2, \ldots, N_b \tag{21}$$

The inner product in (20) or (21) is defined as:

$$\langle f(\boldsymbol{p}), g(\boldsymbol{p})\rangle = \int_D f(\boldsymbol{p})g(\boldsymbol{p})w(\boldsymbol{p})\mathrm{d}\boldsymbol{p} \tag{22}$$

where $f(\boldsymbol{p})$ and $g(\boldsymbol{p})$ denote the arbitrary functions in domain $D$; $w(\boldsymbol{p})$ denotes a multivariate weight function in domain $D$ (usually $w(\boldsymbol{p}) = 1$), which is similar to the univariate weight function introduced in Section III-B; $D = \{\boldsymbol{p} \in \mathbb{R}^l | a_d \le p_d \le b_d, d = 1, 2, \ldots, l\}$ is the definition domain of $\boldsymbol{p}$; and $\mathrm{d}\boldsymbol{p} = \mathrm{d}p_1\mathrm{d}p_2\cdot\ldots\cdot\mathrm{d}p_l$. Noticing that the orthogonal polynomial definition (10) only relies on the definition of inner product, we can define the multivariate orthogonal polynomial from it likewise by just replacing the definition of univariate inner product (7) with the multivariate one (22).

Equation (20) or (21) is an $n \times N_b$ dimensional equation set with respect to $n \times N_b$ unknown coefficients. By solving this equation set, the coefficients and polynomial approximations $x_k^N(\boldsymbol{p})$ $(k = 1, 2, \ldots, n)$ of the function vector $\boldsymbol{x}(\boldsymbol{p})$ are obtained.

## IV. Solving PPs by Polynomial Approximation Based on Galerkin Method

### A. Approximation of Implicit Functions Governed by Parametric NAEs

Consider the parametric NAEs $\boldsymbol{0}=\boldsymbol{f}(\boldsymbol{x},\boldsymbol{p})$ in (1). These equations govern an implicit parameter-state function vector $\boldsymbol{x}(\boldsymbol{p})$ that satisfies $\boldsymbol{f}(\boldsymbol{x}(\boldsymbol{p}),\boldsymbol{p})=\boldsymbol{0}$ over the whole domain of $\boldsymbol{p}$. This function vector can be approximated by polynomial approximation based on Galerkin method.

To reduce the error of approximating $\boldsymbol{x}(\boldsymbol{p})$, the polynomial approximation $\boldsymbol{x}^N(\boldsymbol{p})$ should make each element of $\boldsymbol{f}(\boldsymbol{x}^N(\boldsymbol{p}),\boldsymbol{p})$ close to 0. Thus, the residual function can be

$$R_k(\boldsymbol{p})=f_k(\boldsymbol{x}^N(\boldsymbol{p}),\boldsymbol{p}) \quad k=1,2,\ldots,n \tag{23}$$

Each element $x_k^N(\boldsymbol{p})$ of $\boldsymbol{x}^N(\boldsymbol{p})$ is the linear combination of the trial basis $\{\Phi_i(\boldsymbol{p}), i=1,2,\ldots,N_b\}$, as shown in (18), and its coefficients $\{c_{ki}, i=1,2,\ldots,N_b\}$ are unknown.

With (23) and a chosen test basis, the Galerkin equation (20) or (21) is established, from which unknown coefficients can be solved out and thus an explicit approximation function vector $\boldsymbol{x}^N(\boldsymbol{p})$ is obtained.

Although Galerkin method has several merits such as global approximation and controllable accuracy, it also has some drawbacks in terms of dimensions of the unknowns and equations. As indicated in (18) and (20), the numbers of both Galerkin equations and unknown coefficients are $N_b$ times those of original equations and variables. Furthermore, according to (17), $N_b$ could be very big even though both $N$ and $l$ are moderate values, as shown in Table I. Therefore, the Galerkin equations may have a very large dimension and thus become difficult to solve, which poses a challenge to the method.

TABLE I
Size of Bases in Polynomial Approximation

| $N$ | $l$ | $N_b$ |
|---|---|---|
| 3 | 2 | 10 |
| 3 | 3 | 20 |
| 3 | 4 | 35 |
| 2 | 5 | 21 |
| 3 | 5 | 56 |
| 4 | 5 | 126 |
| 2 | 10 | 66 |
| 3 | 10 | 286 |
| 4 | 10 | 1001 |

Fortunately, on the one hand, although there could be dozens even hundreds of parameters in engineering problems, practical studies rarely involve more than 5 parameters at one time. On the other hand, although $N$ should be a relatively big value to ensure high accuracy, for most problems, 3rd-order approximation often yields quite good results. Therefore, it is not likely for $N_b$ to be much greater than 50, which signifies that the computation burden of Galerkin method is usually acceptable in practical applications.

### B. Approximation of Implicit Functions Governed by Parametric NLP

Consider the parametric NLP (2) that determines an implicit function $\boldsymbol{x}(\boldsymbol{p})$. Similar to the NLP where $\boldsymbol{p}$ is a deterministic value, the parametric NLP can be solved by the interior point method based on the logarithmic barrier function likewise.

First, introduce the vector of parametric slack variables $\boldsymbol{u}(\boldsymbol{p})=[u_1,u_2,\ldots,u_s]\in\mathbb{R}^s$ that satisfies $\boldsymbol{u}(\boldsymbol{p})\geq\boldsymbol{0}$ and $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{p})+\boldsymbol{u}(\boldsymbol{p})=\boldsymbol{0}$, where $\boldsymbol{u}(\boldsymbol{p})$ denotes $\boldsymbol{u}$ is an implicit function of $\boldsymbol{p}$.

Then, formulate the parametric augmented Lagrange function

$$\begin{aligned}L(\boldsymbol{x},\boldsymbol{p})=&f(\boldsymbol{x},\boldsymbol{p})-(\boldsymbol{y}(\boldsymbol{p}))^{\mathrm{T}}\boldsymbol{h}(\boldsymbol{x},\boldsymbol{p})-\\&(\boldsymbol{z}(\boldsymbol{p}))^{\mathrm{T}}(\boldsymbol{g}(\boldsymbol{x},\boldsymbol{p})+\boldsymbol{u}(\boldsymbol{p}))-\mu\sum_{k=1}^{s}\lg(u_k(\boldsymbol{p}))\end{aligned} \tag{24}$$

where $\boldsymbol{y}(\boldsymbol{p})\in\mathbb{R}^r$ and $\boldsymbol{z}(\boldsymbol{p})\in\mathbb{R}^s$ are the parametric Lagrange multipliers for equality and inequality constraints, respectively; and $\mu>0$ is the barrier parameter.

Finally, obtain the NAEs by the KKT condition, namely

$$\begin{cases}\boldsymbol{0}=\boldsymbol{f}_x(\boldsymbol{x},\boldsymbol{p})-(\boldsymbol{y}(\boldsymbol{p}))^{\mathrm{T}}\boldsymbol{h}_x(\boldsymbol{x},\boldsymbol{p})-(\boldsymbol{z}(\boldsymbol{p}))^{\mathrm{T}}\boldsymbol{g}_x(\boldsymbol{x},\boldsymbol{p})\\\boldsymbol{0}=\boldsymbol{h}(\boldsymbol{x},\boldsymbol{p})\\\boldsymbol{0}=\boldsymbol{g}(\boldsymbol{x},\boldsymbol{p})+\boldsymbol{u}(\boldsymbol{p})\\\boldsymbol{0}=\boldsymbol{Z}(\boldsymbol{p})\boldsymbol{U}(\boldsymbol{p})+\mu\boldsymbol{e}\end{cases} \tag{25}$$

where subscript $\boldsymbol{x}$ denotes the Jacobian matrix with respect to $\boldsymbol{x}$; $\boldsymbol{Z}(\boldsymbol{p})$ and $\boldsymbol{U}(\boldsymbol{p})$ are the diagonal matrices composed of vectors $\boldsymbol{z}(\boldsymbol{p})$ and $\boldsymbol{u}(\boldsymbol{p})$, respectively; and $\boldsymbol{e}\in\mathbb{R}^s$ is the vector whose components are all 1.

By utilizing the above formulation, the parametric NLP (2) can be solved by a new method [32] based on polynomial approximation, Galerkin method and interior point method.

On the one hand, for a given $\mu$, (25) determines an implicit function $\boldsymbol{x}(\boldsymbol{p},\mu)$ as well as implicit functions $\boldsymbol{y}(\boldsymbol{p},\mu)$, $\boldsymbol{z}(\boldsymbol{p},\mu)$ and $\boldsymbol{u}(\boldsymbol{p},\mu)$, where $\mu$ is treated as a constant. By employing the polynomial approximation method in Section IV-A, their approximates $\boldsymbol{x}^N(\boldsymbol{p},\mu)$, $\boldsymbol{y}^N(\boldsymbol{p},\mu)$, etc. can be obtained.

On the other hand, $\mu$ is an algorithmic parameter that gradually decreases from an initial value during the iterative process of interior point method. When $\mu$ becomes small enough, the solution $\boldsymbol{x}(\boldsymbol{p},\mu)$ of (25) becomes the optimum solution $\boldsymbol{x}(\boldsymbol{p})$ of (2), and $\boldsymbol{x}^N(\boldsymbol{p},\mu)$ becomes the desired explicit polynomial approximation $\boldsymbol{x}^N(\boldsymbol{p})$ of the parametric NLP. For more details of this new method, please refer to [32].

It should be noted that the method also provides another two useful explicit functions $\boldsymbol{y}^N(\boldsymbol{p})$ and $\boldsymbol{z}^N(\boldsymbol{p})$, which are polynomial approximations of $\boldsymbol{y}(\boldsymbol{p})$ and $\boldsymbol{z}(\boldsymbol{p})$, respectively. The first one can indicate how sensitive the objective function is to the equality constraints as $\boldsymbol{p}$ changes. The second one can be used to identify which inequality constraints are active for different values of $\boldsymbol{p}$.

### C. Approximation of Implicit Functions Governed by Parametric DAEs

Consider the parametric DAEs (3). The essential difference between (3) and (1) is that (1) defines a time-invariant

function $\boldsymbol{x}(\boldsymbol{p})$, whereas (3) defines the time-dependent parametric trajectories $\boldsymbol{x}(t,\boldsymbol{p})$ and $\boldsymbol{y}(t,\boldsymbol{p})$. Therefore, the polynomial approximation (18) of $\boldsymbol{x}(\boldsymbol{p})$ needs to be rebuilt to accommodate $t$. The trick to do this is making expansion coefficients time-variant.

Let $\boldsymbol{x}^N(t,\boldsymbol{p})$ and $\boldsymbol{y}^N(t,\boldsymbol{p})$ be $N^{\text{th}}$-order polynomial approximations of $\boldsymbol{x}(t,\boldsymbol{p})$ and $\boldsymbol{y}(t,\boldsymbol{p})$, respectively, and their elements $x_k^N(t,\boldsymbol{p})$ $(k=1,2,\ldots,n)$ and $y_q^N(t,\boldsymbol{p})$ $(q=1,2,\ldots,m)$ are denoted by

$$\begin{cases} x_k^N(t,\boldsymbol{p}) = \sum_{i=1}^{N_b} c_{ki}^x(t)\Phi_i(\boldsymbol{p}) \\ y_q^N(t,\boldsymbol{p}) = \sum_{i=1}^{N_b} c_{qi}^y(t)\Phi_i(\boldsymbol{p}) \end{cases} \tag{26}$$

where $c_{ki}^x$ and $c_{qi}^y$ are the time-variant coefficients. Correspondingly, the derivative of $x_k$ with respect to time is

$$\frac{\mathrm{d}x_k}{\mathrm{d}t} \approx \frac{\mathrm{d}x_k^N}{\mathrm{d}t} = \sum_{i=1}^{N_b} \Phi_i(\boldsymbol{p}) \frac{\mathrm{d}c_{ki}^x}{\mathrm{d}t} \tag{27}$$

The residual functions are defined by using the idea in Section IV-A, i.e., $\boldsymbol{x}^N(t,\boldsymbol{p})$ and $\boldsymbol{y}^N(t,\boldsymbol{p})$ should make (3) hold as much as possible. Considering (26) and (27), the residual functions of $n$ differntial equations and $m$ algebraic equations are defined as

$$\begin{cases} R_k^f(\boldsymbol{p}) = f_k(\boldsymbol{x}^N,\boldsymbol{y}^N,\boldsymbol{p}) - \sum_{i=1}^{N_b} \Phi_i(\boldsymbol{p}) \frac{\mathrm{d}c_{ki}^x}{\mathrm{d}t} \\ R_q^g(\boldsymbol{p}) = g_q(\boldsymbol{x}^N,\boldsymbol{y}^N,\boldsymbol{p}) \end{cases} \tag{28}$$

where $k=1,2,\ldots,n$; and $q=1,2,\ldots,m$.

By projecting the residual functions onto the space spanned by test basis $\{\Psi_j(\boldsymbol{p})\}$ $(j=1,2,\ldots,N_b)$, which is determined by traditional or generalized Galerkin method, we obtain the following Galerkin equations.

$$\begin{cases} \left\langle f_k(\boldsymbol{x}^N,\boldsymbol{y}^N,\boldsymbol{p}), \Psi_j(\boldsymbol{p}) \right\rangle - \sum_{i=1}^{N_b} \left\langle \Phi_i(\boldsymbol{p}), \Psi_j(\boldsymbol{p}) \right\rangle \frac{\mathrm{d}c_{ki}^x}{\mathrm{d}t} = 0 \\ \left\langle g_q(\boldsymbol{x}^N,\boldsymbol{y}^N,\boldsymbol{p}), \Psi_j(\boldsymbol{p}) \right\rangle = 0 \end{cases} \tag{29}$$

where $k=1,2,\ldots,n$; $q=1,2,\ldots,m$; and $j=1,2,\ldots,N_b$.

Note that $\boldsymbol{p}$ disappears after working out the inner product. Hence, (29) is a set of DAEs with respect to $\{c_{ki}^x\}$ and $\{c_{qi}^y\}$, where $\{c_{ki}^x\}$ acts as $n\times N_b$ state variables and $\{c_{qi}^y\}$ acts as $m\times N_b$ algebraic variables. By solving the DAE set [26], the coefficients $\{c_{ki}^x\}$ and $\{c_{qi}^y\}$ can be obtained, and thus the explicit approximate polynomial solutions $\boldsymbol{x}^N(t,\boldsymbol{p})$ and $\boldsymbol{y}^N(t,\boldsymbol{p})$ for the parametric DAEs are obtained.

### *D. Galerkin Method Combined with Generalized Polynomial Chaos (gPC) for Stochastic Problems*

In addition to PPs, the Galerkin method can also be used to solve stochastic problems where parameters are random variables. Under this circumstance, this method is usually combined with the gPC.

The gPC method [24] is an extension of Wiener's polynomial chaos [36], and has become a popular method for stochastic analysis of complex systems such as fluid dynamics [37] and control engineering [38].

Let $\boldsymbol{x}(\boldsymbol{Z})$ denote model of the stochastic problem, and $\boldsymbol{Z}=[Z_1,Z_2,\ldots,Z_l]$ denote the vector of independent random variables with a joint probability density function (PDF) $\rho(\boldsymbol{Z})=\prod_{i=1}^{l}\rho_i(Z_i)$. The specialty of gPC is choosing univariate orthogonal polynomial sequence with respect to the PDF $\rho_i(Z_i)$ for each $Z_i$ when constructing the polynomial basis. This choice may make the resultant polynomial approximation optimal in the sense of probability measure.

The orthogonal (and also optimal) polynomials corresponding to different probability distributions can be found in the Askey scheme [16], [24], four of which are listed in Table II. For other distributions, the corresponding orthogonal polynomials can be acquired by the Stieltjes or Chebyshev algorithm [39], or these distributions can be transformed to distributions with known orthogonal polynomials in the Askey scheme [25].

TABLE II
SOME PROBABILITY DISTRIBUTIONS AND THEIR CORRESPONDING ORTHOGONAL POLYNOMIALS

| Distribution | Orthogonal polynomial | Support |
|---|---|---|
| Gaussian | Hermite | $(-\infty,\infty)$ |
| Gamma | Laguerre | $[0,\infty)$ |
| Beta | Jacobi | $[a,b]$ |
| Uniform | Legendre | $[a,b]$ |

By combining Galerkin method with gPC, a new method for stochastic problems, called stochastic Galerkin method, is established. This method can be regarded as a special polynomial approximation based on Galerkin method, and thus most contents in previous sections can be generalized to this new method. Only small changes should be made, i.e., replacing the parameters $\boldsymbol{p}$ with random variables $\boldsymbol{Z}$, choosing polynomial basis to be corresponding orthogonal polynomial basis, and letting the weight function in the inner production definition (22) to be the PDF $\rho(\boldsymbol{Z})$. Under this circumstance, the inner product is defined as

$$\left\langle f(\boldsymbol{Z}), g(\boldsymbol{Z}) \right\rangle = \int_S f(\boldsymbol{Z}) g(\boldsymbol{Z}) \rho(\boldsymbol{Z}) \mathrm{d}\boldsymbol{Z} \tag{30}$$

where $f(\boldsymbol{Z})$ and $g(\boldsymbol{Z})$ are two arbitrary functions; and $S$ is the definition domain of $\boldsymbol{Z}$.

Let the $N^{\text{th}}$-order polynomial approximation of the $k^{\text{th}}$ element of the function vector $\boldsymbol{x}(\boldsymbol{Z}):\mathbb{R}^l\to\mathbb{R}^n$ denoted by

$$x_k^N(\boldsymbol{Z}) = \sum_{i=0}^{N_b} c_{ki}\Phi_i(\boldsymbol{Z}) \quad k=1,2,\ldots,n \tag{31}$$

where $\Phi_i(\boldsymbol{Z})$ is the basis function, which is the product of univariate orthogonal polynomial in each dimension. By utilizing the aforementioned Galerkin method, the coefficient $c_{ki}$ and polynomial approximation $x_k^N(\boldsymbol{Z})$ can be obtained.

The probability distribution and statistical characteristics of $\boldsymbol{x}$ can be calculated by the acquired polynomial approximation. The expectation and variance functions of $x_k, k=1,2,\ldots,n$ are

$$\begin{cases} E(x_k) \approx E(x_k^N) = \langle x_k^N, 1 \rangle = c_{k0} \\ Var(x_k) \approx Var(x_k^N) = \langle x_k^N, x_k^N \rangle - E^2(x_k^N) = \sum_{i=1}^{N_b} c_{ki}^2 \chi_i - c_{k0}^2 \end{cases} \tag{32}$$

where $c_{k0}$ is the coefficient of constant term; and $\chi_i = \langle \Phi_i(\boldsymbol{Z}), \Phi_i(\boldsymbol{Z}) \rangle$.

The higher moments and probability distribution functions of $x_k, k = 1, 2, \ldots, n$ can be calculated by the values of the polynomial function $x_k^N(\boldsymbol{Z})$ at many samples of $\boldsymbol{Z}$ generated by MCS or Latin hypercube sampling. This sampling is inexpensive and thus efficient, since it is very simple to compute the values of a polynomial.

## V. Examples of Power System PPs Solved by Galerkin Method

Polynomial approximation based on Galerkin method is a powerful tool to tackle many PPs in power system analysis. This section presents four application examples and some potential applications. These examples can be modeled as three types of typical PPs introduced in Section II, and solved by Galerkin method according to the process introduced in Section IV.

### A. PPF

PPF, introduced by Borkowska [9], aims to analyze the probability uncertainty of system states caused by system parameters with probability uncertainty such as the fluctuation of loads, power outputs of DGs. Its results are valuable for system analysis such as assessing the risk that system states exceed the operational limit.

Mathematically, PPF is the stochastic parametric NAE problem $\boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p})$ described by (1). Here $\boldsymbol{p}$ denotes the vector of uncertain parameters such as stochastic power injections at wind farms, photovoltaic plants, and load nodes; $\boldsymbol{x}$ denotes the vector of system states such as the nodal voltage amplitudes and phase angles (or the real part and imaginary part); and $\boldsymbol{f}(\cdot)$ denotes the power flow equations.

This problem can be tackled by two approaches in the context of polynomial approximation based on Galerkin method. The first approach follows the general solving strategy of PPs in Section IV-A, i.e., first approximating the relationship between state variables and parameters, and then calculating the probability distributions of system states by taking the probability distributions of parameters into account. The second approach combines the Galerkin method and gPC in Section IV-D, i. e., considering the probability distributions of parameters when approximating the parameter-state relationship. The two approaches are basically the same, whereas the latter is better from the perspective of probability measure.

There are a few studies on solving the PPF problem by Galerkin method. In [28], [40] and [41], methods based on the rectangular coordinate and polar coordinate are proposed, respectively. In [29], a method considering the correlation of wind generation is proposed.

Figure 4 extracted from [29] shows the PDF calculation results of the IEEE 30-bus system with two stochastic parameters. The power outputs of wind farms at nodes 18 and 19 are modeled as stochastic parameters following Weibull distribution, and the PDF of voltage amplitude at node 30 is depicted to compare the accuracy of Galerkin method and CM. It can been seen that Galerkin method combined with gPC can achieve a high accuracy almost the same with the MCS, whereas the CM has conspicuous inaccuracy because the linearization technique that the CM is based on cannot reflect the nonlinearity of power flow equations. This result validates that Galerkin method can retain the nonlinearity of power flow equation and thus yields accurate calculation results.

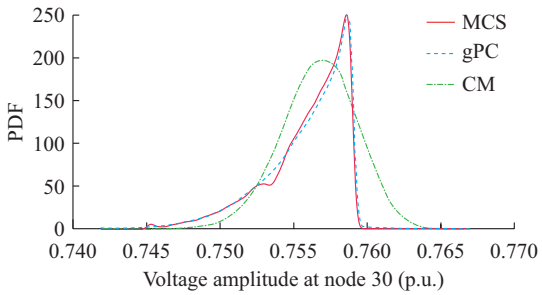


Fig. 4. PDFs of voltage amplitude at node 30 of IEEE 30-bus system obtained by benchmark MCS with $10^5$ samples, 3rd-order Galerkin method combined with gPC, and CM.

### B. Approximation of Local SVSRB

The voltage stability largely depends on the loads, which are regarded as parameters here. The SVSRB is the hypersurface that splits the parameter space into the stable subspace and the unstable one. According to the criteria of static voltage stability, the SVSRBs can be classified into saddle-node bifurcation surfaces and practical security-constrained surfaces, etc.

The saddle-node bifurcation surface can be modeled as:

$$\begin{cases} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{0} \\ \boldsymbol{\eta}^{\mathrm{T}} \boldsymbol{f}_x = \boldsymbol{0} \\ \boldsymbol{\eta}^{\mathrm{T}} \boldsymbol{\eta} = 1 \end{cases} \tag{33}$$

where $\boldsymbol{f}_x$ is Jacobian matrix of power flow equations; and $\boldsymbol{\eta}$ is the left eigenvector that corresponds to the zero eigenvalue of $\boldsymbol{f}_x$. The practical security-constrained surface can be modeled as:

$$\begin{cases} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{0} \\ v_k - \underline{V}_k = 0 \end{cases} \tag{34}$$

where $v_k$ is certain concerned electrical quantity such as the nodal voltage amplitude and generator reactive output; and $\underline{V}_k$ is the corresponding critical value such as minimal bus voltage or maximal reactive power output of generator.

Equation (33) contains $2n + l$ variables and $2n + 1$ equations, where $n$ and $l$ denote the sizes of $\boldsymbol{x}$ and $\boldsymbol{p}$, respectively. Therefore, there are $l - 1$ independent variables, and thus (33) determines an $(l-1)$-variate implicit function $p_1 = g(p_2, p_3, \ldots, p_l)$. Similarly, (34) contains $n + l$ variables and $n +$

1 equations, and thus also determines an $(l-1)$-variate implicit function.

Mathematically, both (33) and (34) belong to parametric nonlinear algebraic problems (1), and thus can be solved by Galerkin method according to Section IV-A. Let the resultant polynomial approximation be denoted by $g^N(p_2,p_3,\ldots,p_l)$, and then the acquired SVSRB can be denoted by the equation $B(\boldsymbol{p})=p_1-g^N(p_2,p_3,\ldots,p_l)=0$.

Figure 5 extracted from [31] shows a two-parameter SVSRB, which is a curve in $\mathbb{R}^2$, where $p_1=P_{L7}$ and $p_2=Q_{L7}$ are the active and reactive power at node 7 of IEEE 10-bus system, respectively. The polynomial approximation constructed by Galerkin method is $p_1=43.02-0.1536p_2-0.00436p_2^2-7.369\times10^{-5}p_2^3$. It can be seen that this polynomial approximation is almost identical to the exact SVSRB (obtained by sampling), whereas the tangent hyperplane and 2nd-order tangent hypersurface (based on the 1st- and 2nd-order Taylor expansions, respectively) have notable error. The accuracy advantage of the Galerkin method roots in its global characteristic compared with the Taylor expansion-based local methods.

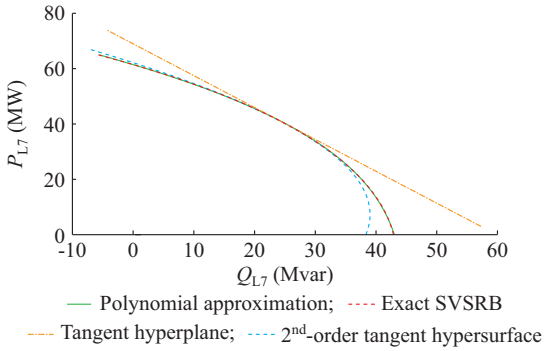


Fig. 5. A two-parameter SVSRB.

## C. Approximation of Global SVSRB

The previous subsection introduces how to acquire the local SVSRB defined by a single stability criterion. In practical power systems, there may exist many stability criteria, each of which corresponds to an SVSRB. A straightforward approach for obtaining the global SVSRB is to calculate these SVSRBs separately and then splice them together. Nevertheless, this approach is computationally intensive and the splicing process is troublesome. This subsection introduces a new approach that obtains the global SVSRB at one go by modeling this problem as the parametric NLP.

With other $l-1$ parameters $(p_2,p_3,\ldots,p_l)$ fixed, increasing $p_1$ until an arbitrary stability criterion is violated, and then the resultant parameter point $\boldsymbol{p}$ is certainly on the global SVSRB, as illustrated by Fig. 6. Therefore, the global SVSRB can be modeled as the parametric NLP problem [32]:

$$\begin{cases}\max\limits_{\boldsymbol{x}} \ p_1 \\ \text{s.t. } \boldsymbol{f}(\boldsymbol{x},p_1;p_2,p_3,\ldots,p_l)=\boldsymbol{0} \\ \quad \boldsymbol{g}(\boldsymbol{x},p_1;p_2,p_3,\ldots,p_l)\le\boldsymbol{0}\end{cases} \tag{35}$$

where $\boldsymbol{g}(\cdot)$ is the vector consisting of all stability constraint functions.

The above parametric NLP problem can be solved by the Galerkin method according to Section IV-B. Let the resultant polynomial approximation denoted by $g^N(p_2,p_3,\ldots,p_l)$, and then the acquired global SVSRB can be denoted by the equation $B(\boldsymbol{p})=p_1-g^N(p_2,p_3,\ldots,p_l)=0$. Furthermore, the Lagrange multipliers corresponding to inequality constraints are also approximated and can be used to identify which inequality constraints are active for different points of parameter $\boldsymbol{p}$ on the global SVSRB.

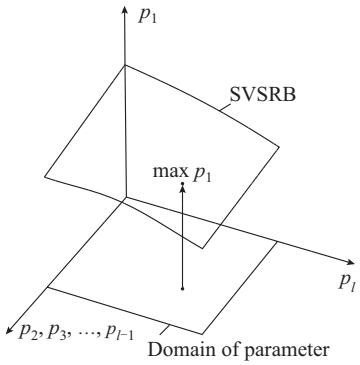


Fig. 6. Optimization model of global SVSRB.

## D. Parametric Time-domain Dynamic Simulation

Apart from power system steady states (e.g., power flow), parameter uncertainty also has great impact on system dynamic states and performance, resulting in the parametric time-domain dynamic simulation problem.

This problem can be modeled as the parametric DAE problem (3) solved by Galerkin method according to Section IV-C, as elaborated in [33], [34]. If parameters are further random variables, Galerkin method can be combined with the gPC in Section IV-D. The solution provides explicit and accurate system dynamic trajectories $\boldsymbol{x}^N(t,\boldsymbol{p})$ and $\boldsymbol{y}^N(t,\boldsymbol{p})$ at each possible value of parameter $\boldsymbol{p}$, and is very useful to analyze the impact of parameters on system dynamic behavior.

Figure 7 extracted from [34] depicts the post-fault trajectories of the difference between power angles of generators 2 and 1 ($\delta_{21}$) in the IEEE 3-generator 9-bus system. The power output of generator 2 ($P_{G2}$) is the studied parameter and varies from 55 MW (rated value) to 135 MW. It can be seen that when $P_{G2}=80$ MW, which varies slightly from the rated value, both the trajectory obtained by Galerkin method and that of the sensitivity method track the exact trajectory well before $t=2$ s. Nevertheless, after that time, the latter deviates obviously from the exact one whereas the former retains its accuracy. When $P_{G2}=135$ MW, which varies largely, the former still has good approximation accuracy for nearly all the time, whereas the latter loses its accuracy since $t=0.5$ s. This result validates that Galerkin method can yield an accurate system dynamic trajectory, whereas the sensitivity method may not.

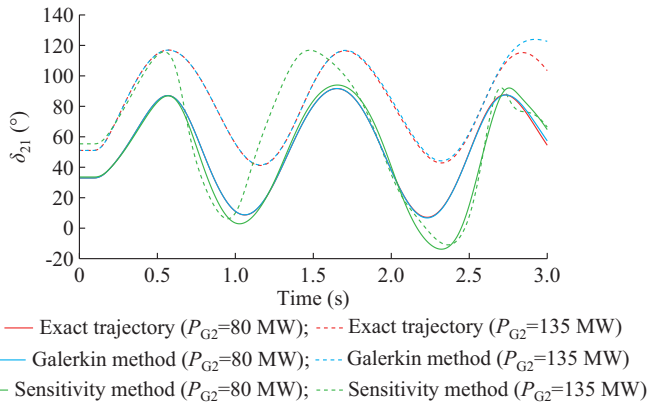


Fig. 7. Exact trajectories of difference between power angles of generators 2 and 1 and those approximated by Galerkin method and sensitivity method for $P_{G2}$=80 MW and $P_{G2}$=135 MW.

### *E. Potential Applications*

In addition to the above applications in published literature, the polynomial approximation based on Galerkin method also has many potential applications. The introduced method provides a polynomial-form surrogate model simplifying the original model, and can be applied to problems that sensitivity method can deal with. Some potential applications on parameter design and DG uncertainty are as follows.

OPF [42] aims to minimize certain cost, e. g., generation cost, by adjusting control variables. The uncertainty of integrated large-scale DGs has necessitated the extension of OPF to stochastic optimal power flow (SOPF). A research perspective for SOPF is to study the impact of uncertainty of DG power outputs (namely parameters) on the optimal solution of OPF. This problem can be modeled as parametric NLP (2), and thus solved by Galerkin method according to Section IV-B. Another perspective is to find the optimal solution under the chance-constraint model [43], and how to apply Galerkin method to this problem is worth exploring.

Power system dynamic control such as model predictive control [44] is a fundamental tool to maintain system security and stability. It is usually not easy to optimize the control model because it involves complicated differential equations representing system dynamic behavior, e. g., transient process, mid-long term process. The Galerkin method can be used to approximate the relationship between control variables (namely parameters) and state variables just like solving (3). With state variables replaced by the acquired polynomial functions of control variables, the control model becomes an explicit model composed of only control variables and their simple functions, and is easy to solve.

Generator parameters, e. g., excitation parameters, have great impact on power system small-disturbance stability and should be optimized. Given a parameter value, a group of system eigenvalues and the stability margin can be calculated, which yields a PP like (33). Galerkin method can be applied to this problem, and results in a polynomial function describing the relationship between parameters and stability, by which parameters can be tuned.

When studying the characteristics of the internal network, it is necessary to perform external network equivalence. A PP can be established by regarding tie-line transmission power as functions of mutable variables (namely parameters) of the external network. Then, Galerkin method can be used to solve this problem, and thus the impact of the external network is equivalent to the acquired polynomial approximation.

Besides, there are some parametric or stochastic problems solved by polynomial approximation based on collocation methods, e. g., available delivery capability assessment considering DG uncertainty [45]. Most of them can also be solved by the introduced Galerkin method.

## VI. Conclusion and Future Work

This paper provides a review of the theory of polynomial approximation based on Galerkin method and its applications in power system PPs as well as stochastic problems.

The PP aims to seek an explicit function describing the relationship between uncertain parameters and system states. The acquired explicit function is significant for studying the impact of uncertain parameters on system states or performances. This paper introduces three types of typical PPs, namely parametric NAEs, NLP, and DAEs. Besides, the stochastic problem is treated as a special case of PPs whose parameters are additionally following certain probability distributions, and thus can be solved by Galerkin method as well.

In terms of mathematical theory, this paper elaborates the introduced method. Galerkin method is a function approximation method that seeks an explicit approximate function by projection technique. Combined with polynomial approximation, Galerkin method can provide the PP with an accurate and quasi-optimal polynomial solution.

In terms of applications, this paper presents some examples of power system PPs, namely PPF, SVSRB, and parametric time-domain dynamic simulation. These examples can be modeled as three types of typical PPs and solved by the introduced method. The acquired solutions are global and remain accurate in strong linearity case compared with those acquired by popular sensitivity methods.

There are many prospective works for the introduced method and its applications in power systems. So far, this method only has a handful of applications in power system analysis. Actually, many more power system problems such as those in Section V-E can be modeled as the three types of typical PPs and thus solved by this method.

Another worthwhile work is to refine this method. The computation burden of this method may become very large when system scale and parameter number are large. Promising approaches for reducing the computation burden include constructing a sparser basis, adopting dimension reduction techniques, etc.

## References

[1] I. A. Hiskens and J. Alseddiqui, “Sensitivity, approximation, and uncertainty in power system dynamic simulation,” *IEEE Transactions on Power Systems*, vol. 21, no. 4, pp. 1808-1820, Nov. 2006.

[2] X. Fang, B.-M. Hodge, F. Li *et al*., “Adjustable and distributionally robust chance-constrained economic dispatch considering wind power uncertainty,” *Journal of Modern Power Systems and Clean Energy*, vol. 7, no. 3, pp. 658-664, May 2019.

[3] B. Shakerighadi, F. Aminifar, and S. Afsharnia, “Power systems wide-

area voltage stability assessment considering dissimilar load variations and credible contingencies," *Journal of Modern Power Systems and Clean Energy*, vol. 7, no. 1, pp. 78-87, Jan. 2019.
[4] Y. Zhou, H. Wu, C. Gu *et al*., "Global optimal polynomial approximation for parametric problems in power systems," *Journal of Modern Power Systems and Clean Energy*, vol. 7, no. 3, pp. 500-511, May 2019.
[5] T. B. Nguyen and M. A. Pai, "Dynamic security-constrained rescheduling of power systems using trajectory sensitivities," *IEEE Transactions on Power Systems*, vol. 18, no. 2, pp. 848-854, May 2003.
[6] H. Choi, P. J. Seiler, and S. V. Dhople, "Propagating uncertainty in power-system DAE models with semidefinite programming," *IEEE Transactions on Power Systems*, vol. 32, no. 4, pp. 3146-3156, Jul. 2017.
[7] Z. Yu and A. Tuzuner, "Wind speed modeling and energy production simulation with Weibull sampling," in *Proceedings of 2008 IEEE PES General Meeting: Conversion and Delivery of Electrical Energy in the 21st Century*, Pittsburgh, USA, Aug. 2008, pp. 1-6.
[8] J. Prada and J. Dorronsoro, "General noise support vector regression with non-constant uncertainty intervals for solar radiation prediction," *Journal of Modern Power Systems and Clean Energy*, vol. 6, no. 2, pp. 268-280, Mar. 2018.
[9] B. Borkowska, "Probabilistic load flow," *IEEE Transactions on Power Apparatus and Systems*, vol. PAS-93, no. 3, pp. 752-759, May 1974.
[10] K. Zou, A. P. Agalgaonkar, K. M. Muttaqi *et al*., "Distribution system planning with incorporating DG reactive capability and system uncertainties," *IEEE Transactions on Sustainable Energy*, vol. 3, no. 1, pp. 112-123, Jan. 2012.
[11] X. Li, X. Zhang, L. Wu *et al*., "Transmission line overload risk assessment for power systems with wind and load-power generation correlation," *IEEE Transactions on Smart Grid*, vol. 6, no. 3, pp. 1233-1242, May 2015.
[12] J. Hammersley, *Monte Carlo Methods*. Beilin: Springer Science & Business Media, 2013.
[13] P. Zhang and S. T. Lee, "Probabilistic load flow computation using the method of combined cumulants and Gram-Charlier expansion," *IEEE Transactions on Power Systems*, vol. 19, no. 1, pp. 676-682, Jan. 2004.
[14] C. S. Saunders, "Point estimate method addressing correlated wind power for probabilistic optimal power flow," *IEEE Transactions on Power Systems*, vol. 29, no. 3, pp. 1045-1054, May 2014.
[15] C. Canuto, M. Y. Hussaini, A. Quarteroni *et al*., *Spectral Methods: Fundamentals in Single Domains*. Beilin: Springer Science & Business Media, 2007.
[16] D. Xiu and G. E. Karniadakis, "The Wiener-Askey polynomial chaos for stochastic differential equations," *SIAM Journal on Scientific Computing*, vol. 24, no. 2, pp. 619-644, Oct. 2002.
[17] D. Funaro, *Polynomial Approximation of Differential Equations*. Berlin: Springer Science & Business Media, 2008.
[18] G. K. Smyth, "Polynomial approximation," in *Wiley StatsRef: Statistics Reference Online*. Hoboken: John Wiley & Sons Ltd., 2014.
[19] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their Q-analogues*. Beilin: Springer Science & Business Media, 2010.
[20] J. P. Keener, *Principles of Applied Mathematics: Transformation and Approximation*. Boca Raton: CRC Press, 2019.
[21] P. G. Constantine, D. F. Gleich, and G. Iaccarino, "Spectral methods for parameterized matrix equations," *SIAM Journal on Matrix Analysis and Applications*, vol. 31, no. 5, pp. 2681-2699, Sept. 2010.
[22] J. Tang, F. Ni, F. Ponci *et al*., "Dimension-adaptive sparse grid interpolation for uncertainty quantification in modern power systems: probabilistic power flow," *IEEE Transactions on Power Systems*, vol. 31, no. 2, pp. 907-919, Mar. 2016.
[23] P. G. Constantine, M. S. Eldred, and E. T. Phipps, "Sparse pseudospectral approximation method," *Computer Methods in Applied Mechanics and Engineering*, vol. 229-232, pp. 1-12, Apr. 2012.
[24] D. Xiu, *Numerical Methods for Stochastic Computations: a Spectral Method Approach*. Princeton: Princeton University Press, 2010.
[25] D. Xiu and G. E. Karniadakis, "Modeling uncertainty in flow simulations via generalized polynomial chaos," *Journal of Computational Physics*, vol. 187, no. 1, pp. 137-167, May 2003.
[26] M. Gerritsma, J.-B. van der Steen, P. Vos *et al*., "Time-dependent generalized polynomial chaos," *Journal of Computational Physics*, vol. 229, no. 22, pp. 8333-8363, Nov. 2010.
[27] X. Wan and G. E. Karniadakis, "An adaptive multi-element generalized polynomial chaos method for stochastic differential equations," *Journal of Computational Physics*, vol. 209, no. 2, pp. 617-642, Nov. 2005.
[28] H. Wu, Y. Zhou, S. Dong *et al*., "Probabilistic load flow by generalized polynomial chaos method," in *Proceedings of 2016 IEEE PES General Meeting*, Boston, USA, Jul. 2016, pp. 1-5.
[29] M. Sun, H. Wu, Y. Qiu *et al*., "Probabilistic load flow for wind power integrated system based on generalized polynomial chaos methods," *Automation of Electric Power Systems*, vol. 41, no. 7, pp. 54-60, Apr. 2017.
[30] Y. Zhou, H. Wu, C. Gu *et al*., "A novel method of polynomial approximation for parametric problems in power systems," *IEEE Transactions on Power Systems*, vol. 32, no. 4, pp. 3298-3307, Jul. 2017.
[31] Y. Qiu, H. Wu, Y. Zhou *et al*., "Global parametric polynomial approximation of static voltage stability region boundaries," *IEEE Transactions on Power Systems*, vol. 32, no. 3, pp. 2362-2371, May 2017.
[32] Y. Qiu, H. Wu, Y. Song *et al*., "Global approximation of static voltage stability region boundaries considering generator reactive power limits," *IEEE Transactions on Power Systems*, vol. 33, no. 5, pp. 5682-5691, Sept. 2018.
[33] L. Li, Y. Qiu, H. Wu *et al*., "Uncertainty analysis of power system time-domain simulation based on generalized polynomial chaos method," in *Proceedings of 2017 IEEE PES General Meeting*, Chicago, USA, Jul. 2017, pp. 1-5.
[34] B. Xia, H. Wu, Y. Qiu *et al*., "A Galerkin method-based polynomial approximation for parametric problems in power system transient analysis," *IEEE Transactions on Power Systems*, vol. 34, no. 2, pp. 1620-1629, Mar. 2019.
[35] L. N. Trefethen, *Approximation Theory and Approximation Practice*. Philadelphia: SIAM, 2019.
[36] N. Wiener, "The homogeneous chaos," *American Journal of Mathematics*, vol. 60, no. 4, pp. 897-936, Oct. 1938.
[37] T. Y. Hou, W. Luo, B. Rozovskii *et al*., "Wiener chaos expansions and numerical solutions of randomly forced equations of fluid mechanics," *Journal of Computational Physics*, vol. 216, no. 2, pp. 687-706, Aug. 2006.
[38] K.-K. K. Kim, D. E. Shen, Z. K. Nagy *et al*., "Wiener's polynomial chaos for the analysis and control of nonlinear dynamical systems with probabilistic uncertainties [historical perspectives]," *IEEE Control Systems Magazine*, vol. 33, no. 5, pp. 58-67, Sept. 2013.
[39] W. Gautschi, "On generating orthogonal polynomials," *SIAM Journal on Scientific and Statistical Computing*, vol. 3, no. 3, pp. 289-317, Sept. 1982.
[40] H. Wu, Y. Zhou, S. Dong *et al*., "Probabilistic load flow based on generalized polynomial chaos," *IEEE Transactions on Power Systems*, vol. 32, no. 1, pp. 820-821, Jan. 2017.
[41] Y. Sun, R. Mao, Z. Li *et al*., "Constant Jacobian matrix-based stochastic Galerkin method for probabilistic load flow," *Energies*, vol. 9, no. 3, p. 153, Mar. 2016.
[42] J. Lavaei and S. H. Low, "Zero duality gap in optimal power flow problem," *IEEE Transactions on Power Systems*, vol. 27, no. 1, pp. 92-107, Feb. 2012.
[43] M. Perninge and C. Hamon, "A stochastic optimal power flow problem with stability constraints – Part II: the optimization problem," *IEEE Transactions on Power Systems*, vol. 28, no. 2, pp. 1849-1857, May 2013.
[44] Y. Qiu, J. Lin, F. Liu *et al*., "Explicit MPC based on the Galerkin method for AGC considering volatile generations," *IEEE Transactions on Power Systems*, vol. 35, no. 1, pp. 462-473, Jan. 2020.
[45] H. Sheng and X. Wang, "Applying polynomial chaos expansion to assess probabilistic available delivery capability for distribution networks with renewables," *IEEE Transactions on Power Systems*, vol. 33, no. 6, pp. 6726-6735, Nov. 2018.

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